On a Method for Determining the Non-Stationary State of Heat in an Ellipsoid.

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Introduction.

1. The first writer, who attempted, with some success, the problem of the determination of the non-stationary state of heat in an *ellipsoid with three unequal axes*, was E. Mathieu,* who showed how the problem could be reduced to the solution of certain ordinary linear differential equations. But he found these equations to be so unmanageable that he contented himself with approximating to their solutions for the spécial case of an *ellipsoid of revolution.* Prof. C. Niven improved upon the results of Mathieu in certain respects in an interesting memoir,† entitled "On the Conduction of Heat in Ellipsoids of Revolution."

In the present paper, I propose (1) to obtain the chief results of Prof. Niven by an entirely different method, and (2), to show how this method can be applied to the case of the ellipsoid with three unequal axes, to obtain similar results which are believed to be new. It may be noted here that, in Art. 6, I point out a mistake in Prof. Niven's memoir.

Preliminary Remarks and Definitions.

2. Let the initial temperature of the ellipsoid be $f(x, y, z)$ and let its boundary, viz., $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, be maintained at temperature zero. Then the required temperature $V(x, y, z, t)$ is such that

$$\frac{\partial V}{\partial t}=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}, \tag{1}$$

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* *Cours de physique mathématique*, Ch. IX.

† *Phil. Trans.*, Vol. CLXXI (1880).

the units being so chosen that the diffusivity is 1,

$$V=0 \qquad \text{on the boundary,} \tag{2}$$

$$V=f(x, y, z) \text{ when } t=0. \tag{3}$$

Thus V can be expressed as a sum of terms of the form $Ae^{-\lambda^2 t}W(x, y, z)$, where the normal function W satisfies the equation

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}=-\lambda^2 W,$$

and vanishes on the boundary; and the constants A are so chosen that the initial condition (3) is satisfied, so that $f(x, y, z)=\Sigma AW$.

When $a=b=c$ so that the ellipsoid becomes a sphere of radius a, an appropriate normal function is

$$S_n(\lambda r)P_n^m(\cos\theta)\cos m\phi,$$

where

$$S_n(x)=(-1)^n\sqrt{\frac{\pi}{2}}\,x^{-\frac{1}{2}}J_{n+\frac{1}{2}}(x), \quad P_n^m(\cos\theta)=\sin^m\theta\frac{d^m P_n}{d\mu^m}, \ \mu \text{ being } \cos\theta,$$

and λ is a root of the equation $S_n(\lambda a)=0$. Throughout the present paper I will represent by W_n^m the normal function corresponding to the function $S_n(\lambda r)P_n^m(\cos\theta)\cos m\phi$, and denote W_n^0 by W_n.

I proceed now to obtain the functions W of various types.

W_0 *For Ellipsoid of Revolution.*

3. Let e denote the eccentricity of the ellipsoid, then neglecting e^4 and higher powers, the equation of the ellipsoid can be written as

$$r=a\{1-\tfrac{1}{3}e^2+\tfrac{1}{3}e^2P_2(\cos\theta)\}, \quad \textit{i. e.,} \quad r=\alpha(1+\varepsilon P_2(\cos\theta)),$$

where $\alpha=a(1-\frac{1}{3}e^2)$ and $\varepsilon=\frac{1}{3}e^2$. Now assume that

$$W_0=S_0(\lambda r)+\varepsilon\sum_{t=1}^{\infty}M_tS_t(\lambda r)P_t(\cos\theta),$$

M_t being an unknown constant to be determined. Then, evidently, W_0 satisfies the partial differential equation (I); and to satisfy the boundary condition we must have

$$0=S_0(\lambda\alpha)+\varepsilon\alpha\frac{\partial S_0(\lambda\alpha)}{\partial\alpha}P_2+\varepsilon\sum_{t=1}^{\infty}M_tS_t(\lambda\alpha)P_t(\cos\theta),$$

since ε^2 and higher powers are neglected. This equation must hold for all

values of $\cos\theta$. Therefore, equating to zero the coefficients of the various zonal harmonics, we get

$$S_0(\lambda\alpha)=0, \tag{1}$$

$$M_2 S_2(\lambda\alpha)+\alpha\frac{\partial S_0(\lambda\alpha)}{\partial\alpha}=0, \tag{2}$$

and all the other M's are zero. Hence the required expression for W_0, in terms of a and e, is

$$W_0=S_0(\lambda r)-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2(\cos\theta),$$

where λ is given by the equation (1). But the general solution of the equation (1) is known to be

$$\lambda\alpha=i\pi,$$

i being any integer. Hence

$$\lambda a=i\pi+\tfrac{1}{3}i\pi e^2.$$

4. In order to obtain a closer approximation to W_0, we will retain e^4 and neglect the sixth and higher powers. Thus the equation of the ellipsoid is

$$r=a[(1-\tfrac{1}{3}e^2-\tfrac{2}{15}e^4)+(\tfrac{1}{3}e^2+\tfrac{1}{21}e^4)P_2+\tfrac{3}{35}e^4P_4],\ \textit{i. e.},\ r=\beta[1+\sigma P_2+\tau P_4],$$

where $\beta=a(1-\frac{1}{3}e^2-\frac{2}{15}e^4)$, $\sigma=\frac{1}{3}e^2+\frac{10}{63}e^4$, and $\tau=\frac{3}{35}e^4$.

Let us assume that

$$W_0=S_0(\lambda r)-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2(\cos\theta)+\tau\sum_{t=1}^{\infty}N_tS_t(\lambda r)P_t(\cos\theta),$$

N_t being an unknown constant to be determined. Expanding by Taylor's Theorem, we have

$$S_0(\lambda r)=S_0(\lambda\beta)+\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}(\sigma P_2+\tau P_4)+\frac{1}{2!}\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}(\sigma P_2+\tau P_4)^2+\ldots,$$

and
$$S_2(\lambda r)=S_2(\lambda\beta)+\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}(\sigma P_2+\tau P_4)+\ldots,$$

when
$$r=\beta(1+\sigma P_2+\tau P_4).$$

Again
$$(P_2)^2=\tfrac{18}{35}P_4+\tfrac{2}{7}P_2+\tfrac{1}{5}.$$

Hence we must have

$$0=S_0(\lambda\beta)+\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}(\sigma P_2+\tau P_4)+\frac{\sigma^2}{2}\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}[\tfrac{18}{35}P_4+\tfrac{2}{7}P_2+\tfrac{1}{5}]$$
$$-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda\beta)P_2-\tfrac{1}{3}e^2\sigma\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}[\tfrac{18}{35}P_4+\tfrac{2}{7}P_2+\tfrac{1}{5}]$$
$$+\tau\sum_{t=1}^{\infty}N_tS_t(\lambda\beta)P_t.$$

This equation must hold for all values of $\cos\theta$. Therefore, equating to zero the coefficients of the various zonal harmonics, we get

$$S_0(\lambda\beta)+\tfrac{1}{10}\sigma^2\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}-\tfrac{1}{15}e^2\sigma\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}=0, \tag{1}$$

$$\tau N_2S_2(\lambda\beta)+\sigma\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}+\tfrac{1}{7}\sigma^2\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot S_2(\lambda\beta)$$
$$-\tfrac{2}{21}e^2\sigma\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}=0, \tag{2}$$

$$\tau N_4S_4(\lambda\beta)+\tau\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}+\tfrac{9}{35}\sigma^2\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}$$
$$-\tfrac{6}{35}e^2\sigma\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}=0; \tag{3}$$

and all the other N's are zero. On substituting the values of β, σ and τ in terms of a and e, the above equations take the forms

$$S_0(\lambda a)-(\tfrac{1}{3}e^2+\tfrac{2}{15}e^4)a\frac{\partial S_0(\lambda a)}{\partial a}+\tfrac{1}{15}e^4a^2\frac{\partial^2 S_0(\lambda a)}{\partial a^2}$$
$$-\tfrac{1}{45}e^4\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\,a\,\frac{\partial S_2(\lambda a)}{\partial a}=0, \tag{4}$$

$$\tfrac{3}{35}N_2S_2(\lambda a)+\tfrac{1}{21}a\frac{\partial S_0(\lambda a)}{\partial a}-\tfrac{2}{21}a^2\frac{\partial^2 S_0(\lambda a)}{\partial a^2}+\tfrac{5}{63}\cdot\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot a\frac{\partial S_2(\lambda a)}{\partial a}=0, \tag{5}$$

$$\tfrac{3}{35}N_4S_4(\lambda a)+\tfrac{3}{35}a\frac{\partial S_0(\lambda a)}{\partial a}+\tfrac{1}{35}a^2\frac{\partial^2 S_0(\lambda a)}{\partial a^2}-\tfrac{2}{35}\cdot\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot a\frac{\partial S_2(\lambda a)}{\partial a}=0. \tag{6}$$

Hence the required expression for W_0 is

$$W_0 = S_0(\lambda r) - \tfrac{1}{3} e^2 \frac{a \dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)} S_2(\lambda r) P_2(\cos\theta) + \tfrac{3}{35} e^4 N_2 S_2(\lambda r) P_2(\cos\theta) + \tfrac{3}{35} e^4 N_4 S_4(\lambda r) P_4(\cos\theta),$$

the values of N_2 and N_4 being given by the equations (5) and (6), and the value of λ being determined from the equation (4).

5. I will now proceed with the solution of the equation (4) of the preceding article. If we neglect e altogether, the equation reduces to $S_0(\lambda a) = 0$, whose roots are given by $\lambda a = i\pi$. Therefore let the full value of λa be

$$i\pi + l_1 e^2 + l_2 e^4 + \ldots,$$

l_1, l_2 being unknown quantities which are to be determined. S_n can be expanded in a series containing a finite number of terms so that

$$S_n(x) = \left[\frac{1}{x} - \frac{n'(n'-1')}{1.2} \cdot \frac{1}{2^2 x^3} + \frac{n'(n'-1')(n'-2')(n'-3')}{1.2.3.4} \cdot \frac{1}{2^4 x^5} - \ldots\right] \sin\left(x + \frac{n\pi}{2}\right)$$
$$+ \left[\frac{n'}{1} \cdot \frac{1}{2x^2} - \frac{n'(n'-1')(n'-2')}{1.2.3} \cdot \frac{1}{2^3 x^4} + \ldots\right] \cos\left(x + \frac{n\pi}{2}\right),$$

where n' stands for $n(n+1)$ and $(n'-t')$ for $(n-t)(n+t+1)$. Hence we have the following:

$$S_0(\lambda a) = \frac{l_1 e^2}{i\pi} \cos i\pi + \left(\frac{l_2}{i\pi} - \frac{l_1^2}{i^2\pi^2}\right) \cos i\pi \cdot e^4 + \ldots,$$

$$S_2(\lambda a) = -\frac{3}{i^2\pi^2} \cos i\pi + \ldots, \quad a \frac{\partial S_0(\lambda a)}{\partial a} = \cos i\pi - \frac{l_1}{i\pi} \cos i\pi \cdot e^2 + \ldots,$$

$$a^2 \frac{\partial^2 S_0(\lambda a)}{\partial a^2} = -2 \cos i\pi + \ldots, \quad a \frac{\partial S_2(\lambda a)}{\partial a} = -\left(1 - \frac{9}{i^2\pi^2}\right) \cos i\pi + \ldots,$$

$$a \frac{\partial S_0(\lambda a)}{\partial a} \cdot a \frac{\partial S_2(\lambda a)}{\partial a} \Big/ S_2(\lambda a) = \tfrac{1}{3}(i^2\pi^2 - 9) \cos i\pi + \ldots.$$

Therefore, substituting the above expressions for $S_0(\lambda a)$, etc., in the equation (4) of the preceding article, and equating to zero the coefficients of e^2 and e^4, we obtain finally

$$\lambda a = i\pi + \tfrac{1}{3} i\pi e^2 + \frac{i\pi}{135}(i^2\pi^2 + 27) e^4 + \ldots.$$

Comparison with the Results of Niven.

6. By neglecting e^6 and higher powers, Prof. C. Niven has obtained an expression * for the parameter λ, viz.:

$$\lambda a = i\pi + \tfrac{1}{3} i\pi e^2 + \frac{i\pi}{405}(i^2\pi^2+27)e^4 + \ldots,$$

which differs from that obtained by me in the preceding article only in so far as the coefficient of e^4 is concerned. After carefully going through Prof. Niven's calculations, I find that the mistake of Prof. Niven must be attributed to some inadvertence on his part, as, by repeating his process, I get the correct result.

From the identity of the values of λ, it follows at once that my expression for W_0 and Prof. Niven's must be identical. For, as is well-known, for the same value of λ there can not be two different solutions of the equation

$$\frac{\partial^2 W_0}{\partial x^2} + \frac{\partial^2 W_0}{\partial y^2} + \frac{\partial^2 W_0}{\partial z^2} + \lambda^2 W_0 = 0,$$

both of which vanish on the boundary.

W_n For Ellipsoid of Revolution, $n>0$.

7. Let e^4 and higher powers be neglected, so that the equation of the ellipsoid is the same as in Art. 3, viz.: $r=\alpha\{1+\varepsilon P_2(\cos\theta)\}$. Then assume that

$$W_n = S_n(\lambda r)P_n(\cos\theta) + \varepsilon \sum_{t=0}^{\infty}{}' H_t S_t(\lambda r) P_t(\cos\theta),$$

where the H's are unknown constants, to be determined, and $\sum\limits_{t=0}^{\infty}{}'$ refers to all the values of t except $t=n$. Thus it is evident that W_n satisfies the partial differential equation (I), and it remains to find the values of the constants H's so as to satisfy the boundary condition. Since ε^2 and higher powers of ε are to be neglected, we have, on putting $r=\alpha(1+\varepsilon P_2)$,

$$S_n(\lambda r) = S_n(\lambda\alpha) + \varepsilon\alpha \frac{\partial S_n(\lambda\alpha)}{\partial \alpha} P_2;$$

also, we have

$$P_2 \cdot P_n = B_{n+2}P_{n+2} + C_n P_n + D_{n-2}P_{n-2},$$

where

$$B_{n+2} = \tfrac{3}{2}\cdot\frac{(n+1)(n+2)}{(2n+3)(2n+1)}, \quad C_n = \frac{n(n+1)}{(2n+3)(2n-1)},$$

$$D_{n-2} = \tfrac{3}{2}\cdot\frac{n(n-1)}{(2n+1)(2n-1)}.$$

* *Loc. cit.*, p. 145.

Hence we must have

$$0=S_n(\lambda\alpha)P_n+\varepsilon\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}[B_{n+2}P_{n+2}+C_nP_n+D_{n-2}P_{n-2}]+\varepsilon\sum_{t=0}^{\infty}{}'H_tS_t(\lambda\alpha)P_t.$$

This must be true for all values of $\cos\theta$. Therefore equating to zero the coefficients of the various zonal harmonics, we get

$$S_n(\lambda\alpha)+\varepsilon\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}C_n\quad=0, \tag{1}$$

$$H_{n+2}S_{n+2}(\lambda\alpha)+\ \alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}B_{n+2}=0, \tag{2}$$

$$H_{n-2}S_{n-2}(\lambda\alpha)+\ \alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}D_{n-2}=0, \tag{3}$$

and all the other H's are zero. Thus the unknown constants are determined and the required expression for W_n, in terms of a and e, is

$$W_n=S_n(\lambda r)P_n-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_n(\lambda a)}{\partial a}}{S_{n+2}(\lambda a)}S_{n+2}(\lambda r)P_{n+2}B_{n+2}$$
$$-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_n(\lambda a)}{\partial a}}{S_{n-2}(\lambda a)}S_{n-2}(\lambda r)P_{n-2}D_{n-2};$$

λ being a root of the equation (1).

But, in terms of a and e, this equation is

$$S_n(\lambda a)-\frac{n^2+n-1}{(2n+3)(2n-1)}e^2a\frac{\partial S_n(\lambda a)}{\partial a}=0. \tag{4}$$

Therefore, if x be a root of the equation $S_n(x)=0$, the corresponding solution of (4) is given by

$$\lambda a=x\left\{1+\frac{n^2+n-1}{(2n+3)(2n-1)}e^2\right\}.$$

W_n^m For Ellipsoid of Revolution.

8. Let e^4 and higher powers be neglected, and assume that

$$W_n^m=S_n(\lambda r)P_n^m(\cos\theta)\cos m\phi+\varepsilon\sum_{t=m}^{\infty}{}'I_t^mS_t(\lambda r)P_t^m(\cos\theta)\cos m\phi,$$

where $\sum_{t=m}^{\infty}{}'$ refers to all values of t from m up to infinity except the value $t=n$, and I_t^m is an unknown constant to be determined. Then it is evident that W_n^m

satisfies the partial differential equation (I). Now, putting $r=\alpha(1+\varepsilon P_2)$, we get

$$S_n(\lambda r)=S_n(\lambda\alpha)+\varepsilon\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}P_2;\ \text{and}\ P_2\cdot P_n^m=B_{n+2}^mP_{n+2}^m+C_n^mP_n^m+D_{n-2}^mP_{n-2}^m,$$

where

$$B_{n+2}^m=\tfrac{3}{2}\cdot\frac{(n-m+1)(n-m+2)}{(2n+3)(2n+1)},\quad C_n^m=\frac{n(n+1)-3m^2}{(2n+3)(2n-1)},$$

$$D_{n-2}^m=\tfrac{3}{2}\cdot\frac{(n+m)(n+m-1)}{(2n+1)(2n-1)}.$$

Hence, from the boundary condition, we must have

$$0=S_n(\lambda\alpha)P_n^m\cos m\phi+\varepsilon\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}[B_{n+2}^mP_{n+2}^m+C_n^mP_n^m+D_{n-2}^mP_{n-2}^m]$$
$$\times\cos m\phi+\varepsilon\sum_{t=m}^{\infty}{}' I_t^mS_t(\lambda\alpha)P_t^m\cos m\phi.$$

Therefore, equating to zero the coefficients of the various surface harmonics, we get

$$S_n(\lambda\alpha)+\varepsilon\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}C_n^m=0, \tag{1}$$

$$I_{n+2}^mS_{n+2}(\lambda\alpha)+\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}B_{n+2}^m=0, \tag{2}$$

$$I_{n-2}^mS_{n-2}(\lambda\alpha)+\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}D_{n-2}^m=0, \tag{3}$$

and all the other I_t^m's are zero.

Thus the required expression for W_n^m in terms of a and e is

$$W_n^m=\left[S_n(\lambda r)P_n^m(\cos\theta)-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_n(\lambda a)}{\partial a}}{S_{n+2}(\lambda a)}S_{n+2}(\lambda r)P_{n+2}^m(\cos\theta)B_{n+2}^m\right.$$
$$\left.-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_n(\lambda a)}{\partial a}}{S_{n-2}(\lambda a)}S_{n-2}(\lambda r)P_{n-2}^m(\cos\theta)D_{n-2}^m\right]\cos m\phi,$$

where λ is a root of the equation (1). But, expressed in terms of a and e, this equation becomes

$$S_n(\lambda a)-\frac{(n^2+n-1)+m^2}{(2n+3)(2n-1)}e^2a\frac{\partial S_n(\lambda a)}{\partial a}=0. \tag{4}$$

Therefore, we obtain

$$\lambda a=x\left\{1+\frac{(n^2+n-1)+m^2}{(2n+3)(2n-1)}e^2\right\},$$

corresponding to the root x of $S_n(x)=0$.

W_0 *For Ellipsoid with Three Unequal Axes.*

9. Let e_1 and e_2 be the eccentricities of the two principal diametral sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $a>b>c$, by planes passing through the major axis. Then, neglecting the fourth and higher powers of e_1 and e_2, the equation to the ellipsoid is

$$r=a[1-\tfrac{1}{6}(e_1^2+e_2^2)(P_0-P_2)+\tfrac{1}{12}(e_2^2-e_1^2)P_2^2\cos 2\phi],$$

i. e.,

$$r=\gamma(1+\varepsilon_1 P_2+\varepsilon_2 P_2^2\cos 2\phi),$$

where $\gamma=a\{1-\frac{1}{6}(e_1^2+e_2^2)\}$, $\varepsilon_1=\frac{1}{6}(e_1^2+e_2^2)$, and $\varepsilon_2=\frac{1}{12}(e_2^2-e_1^2)$.

Now assume that

$$W_0=S_0(\lambda r)+\sum_{t=m}^{\infty}{}'\sum_{m=0}^{\infty}R_{m,t}S_t(\lambda r)P_t^m(\cos\theta)\cos m\phi,$$

where $R_{m,t}$ is an unknown constant to be determined and $\sum_{t=m}^{\infty}{}'$ refers to all values of t from m up to ∞, except $t=0$. Then it is evident that W_0 satisfies the partial differential equation (I). To satisfy the boundary condition, we must have

$$0=S_0(\lambda\gamma)+\gamma\frac{\partial S_0(\lambda\gamma)}{\partial\gamma}(\varepsilon_1 P_2+\varepsilon_2 P_2^2\cos 2\phi)+\sum_{t=m}^{\infty}{}'\sum_{m=0}^{\infty}R_{m,t}S_t(\lambda\gamma)P_t^m(\cos\theta)\cos m\phi,$$

the unknown constants R's being assumed to be of the same order as ε_1 and ε_2. Therefore, equating to zero the coefficients of the various surface harmonics, we get

$$S_0(\lambda\gamma)=0, \tag{1}$$

$$R_{0,2}S_2(\lambda\gamma)+\varepsilon_1\gamma\frac{\partial S_0(\lambda\gamma)}{\partial\gamma}=0, \tag{2}$$

$$R_{2,2}S_2(\lambda\gamma)+\varepsilon_2\gamma\frac{\partial S_0(\lambda\gamma)}{\partial\gamma}=0; \tag{3}$$

and all the other R's are zero. Thus the R's are determined, and we get finally the required expression for W_0 to be

$$W_0=S_0(\lambda r)-\tfrac{1}{6}(e_1^2+e_2^2)\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2(\cos\theta)$$

$$+\tfrac{1}{12}(e_1^2-e_2^2)\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2^2(\cos\theta)\cos 2\phi,$$

where λ is a root of the equation (1). But the roots of the equation (1) are given by

$$\lambda\gamma = i\pi.$$

Hence, in terms of a and the eccentricities, we have

$$\lambda a = i\pi + \tfrac{1}{6}(e_1^2 + e_2^2)\, i\pi.$$

Conclusion.

10. I conclude this paper by pointing out that the results or Arts. 3–9 admit of extension and generalization in various directions. For example, a procedure similar to that of Art. 9 will give us W_n^m for the ellipsoid with three unequal axes. Also, denoting by $\overline{W}_n^m$, the normal function corresponding to

$$S_n(\lambda r) P_n^m(\cos\theta) \sin m\phi,$$

it is obvious that, for the ellipsoid of revolution as well as for the ellipsoid with three unequal axes, $\overline{W}_n^m$ can be obtained in the same way as W_n^m.

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